Non-Perturbative Extension of Cornish-Fisher Value-at-Risk via Dilute Instanton Gas

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Abstract

The Cornish-Fisher expansion (Cornish & Fisher, 1937; Fisher & Cornish, 1960) provides a perturbative quantile expansion of a distribution in terms of its Gaussian quantile and standardised co-moments G^2, G^3, G^4. The expansion recovers the Gaussian Value-at-Risk (RiskMetrics; Longerstaey, 1996) at zeroth order and accommodates skew and kurtosis corrections at second and third order. We observe that the perturbative series fails in the regime where the underlying density admits competing local minima—a configuration that arises naturally for portfolio loss densities under bistable correlation potentials — because tunneling between the metastable states is non-analytic in the variance and contributes no terms to any finite order of the Cornish-Fisher expansion. We provide the non-perturbative correction using Coleman's (1977) dilute-instanton-gas formalism. The partition function decomposes as $Z = Z_{\text{pert}} + \Sigma_{n=1} Z_{\text{n-inst}}$; the combined estimator reads $\text{VaR}_\alpha = (1-p_{\text{crash}}) \cdot \text{VaR}_{\text{CF}} + p_{\text{crash}} \cdot \text{VaR}_{\text{inst}}$ with $p_{\text{crash}} = \exp(-S_{\text{inst}}/\sigma^2)$ and recovers the Gaussian limit as $S_{\text{inst}} \to \infty$. We describe the calibration protocol via the implied risk-neutral density (Breeden & Litzenberger, 1978) and historical cumulant estimators (Martellini & Ziemann, 2010), and pre-register the breach-rate validation protocol per Harvey–Liu–Zhu (2016). A first pre-registered historical settlement of that protocol (June 2026) found the estimator better calibrated than the Gaussian benchmark—breach rate inside the binomial acceptance region where the Gaussian falls outside—but below the Harvey significance bar ($t = 2.14 < 3.0$), with the instanton leg inert under the settlement calibration; the estimator therefore remains a shadow metric and position sizing stays on the classical formula (§8.1).

1. Introduction

The Gaussian Value-at-Risk estimator, $\text{VaR}_\alpha(T) = z_\alpha \cdot \sigma \cdot \sqrt{T}$, remains the workhorse benchmark of regulatory and operational risk reporting (Basel Committee, 2019; Longerstaey, 1996). Its primary failure mode is well-documented: in the presence of heavy tails or skew, the estimator systematically under-states losses (Embrechts *et al.*, 1997; Daníelsson *et al.*, 2001). The Cornish-Fisher (1937) expansion addresses this by augmenting the Gaussian quantile with skew and excess-kurtosis corrections:

```
zα(S, K) = zα
           + (1/6)(zα² − 1) S
           + (1/24)(zα³ − 3zα) K
           − (1/36)(2zα³ − 5zα) S²

VaR_CF(α, T) = zα(S, K) · σ · √T
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where $S = G^3_{iii}/\sigma^3$ is the standardised skew and $K = G^4_{iiii}/\sigma^4 - 3$ is the excess kurtosis. The expansion is convergent for moderate departures from normality and recovers the Gaussian quantile at zeroth order. For portfolio-level losses the relevant tensors are the co-skewness G^3_{ijk} and co-kurtosis G^4_{ijkl}, contracted against the weight vector (Martellini & Ziemann, 2010).

We identify a regime where the Cornish-Fisher expansion is structurally inadequate: when the joint density of correlation-state and portfolio-loss admits two or more local minima separated by a saddle, the tail probability is dominated by a non-analytic contribution from tunneling between the basins. This contribution is invisible to any finite truncation of a perturbative series in the cumulants because it scales as $\exp(-S/\sigma^2)$ and is identically zero for every finite Taylor coefficient in σ.

Contribution. We provide the non-perturbative correction to the Cornish-Fisher expansion via the Coleman (1977) dilute-instanton-gas formula. The partition function decomposes exactly as $Z = Z_{\text{pert}} + \Sigma_{n=1}\, Z_{n\text{-inst}}$; truncated at the perturbative G^4 and single-instanton level, the corresponding VaR estimator is $\text{VaR}_{\text{CF-inst}} = (1-p_{\text{crash}})\, \text{VaR}_{\text{CF}} + p_{\text{crash}}\, \text{VaR}_{\text{inst}}$. The estimator recovers the Gaussian limit (Longerstaey, 1996) when $S = K = 0$ and $S_{\text{inst}} \to \infty$, and the classical Cornish-Fisher quantile in the non-tunneling limit alone. The non-perturbative contribution becomes dominant when the implied correlation density approaches a saddle of the underlying bistable potential.

2. Theoretical Framework

2.1 Density-functional view of VaR

For a portfolio with weights w over N assets, the loss density at horizon T is the marginal of a configuration-space density $\rho(x,T)$ projected onto the directional axis $w^T x$. The α-quantile VaR is then a functional of the full density:

```
VaRα(T) = −Qα(wᵀx | ρ(·,T))
        = −inf { x : ∫₋∞ˣ ρw(y, T) dy ≥ α }
```

The Gaussian VaR is then the quantile of a Gaussian approximation to ρ_w; the Cornish-Fisher VaR is the quantile of the Gram-Charlier expansion to fourth cumulant; and the non-perturbative VaR proposed here is the quantile of a density that decomposes into a perturbative Gaussian-plus-cumulant component and a tunneling-derived second mode.

2.2 Bistable correlation potential

In high-dimensional asset-correlation space, financial systems exhibit metastability: a "normal-correlation" basin in which inter-asset correlations sit near long-run historical averages, and a "crisis-correlation" basin in which correlations approach unity (Pagan & Schwert, 1990; Forbes & Rigobon, 2002). The system spends most of its time in the normal basin; transitions to the crisis basin are rare but produce the bulk of realised tail losses.

We parametrise this with a one-dimensional collective correlation coordinate ρ and a quartic double-well potential:

```
V(ρ) = (λ/4) · (ρ − ρn)² · (ρ − ρc)²
```

with ρ_n the normal-basin centre, $\rho_c > \rho_n$ the crisis-basin centre, and λ the well-separation parameter. The shape is generic for order-parameter dynamics with a quadratic-quadratic interaction and matches the structure used by Coleman (1977) for false-vacuum decay.

2.3 Partition function decomposition

The transition amplitude for the correlation order parameter under the bistable potential admits the path-integral representation

```
Z(T) = ∫ Dρ exp(−S[ρ] / σ²)

S[ρ] = ∫₀ᵀ dt [ (1/2) (dρ/dt)² + V(ρ) ]
```

(In the Euclidean / imaginary-time formulation appropriate for tunneling.) Coleman's result is that for two-well potentials of this form, the partition function decomposes into a perturbative contribution Z_{pert} around either of the two stable minima and a sum of instanton (bounce) contributions $Z_{n\text{-inst}}$ connecting them:

```
Z(T) = Zpert + Σn=1^∞ Zn-inst(T)

Z1-inst ~ T · Kfluct^(−1/2) · exp(−Sinst / σ²)
```

The bounce action S_{inst} is the action of the classical Euclidean trajectory that starts at ρ_n, climbs over the barrier at the saddle, and returns. For the quartic potential above with barrier height ΔV, the result is $S_{\text{inst}} = (2\sqrt{2}/3)\cdot\sqrt{\lambda}\cdot|\rho_c - \rho_n|^3$. This contribution is non-analytic in σ: it cannot be obtained from any finite-order Taylor expansion in the cumulants.

> **Tunneling probability.** Truncated at the single-instanton (dilute-gas) level, the probability of a transition to the crisis basin within horizon T is $p_{\text{crash}}(T) = T \cdot \omega \cdot \exp(-S_{\text{inst}}/\sigma^2)$, where ω is a prefactor depending on the curvature of the false-vacuum well and the saddle. For analytical tractability we collapse the prefactor into a normalisation that is fixed by the calibration protocol of §4.

3. Combined Estimator

3.1 Decomposition by exclusion principle

Within horizon T the system is in exactly one of two mutually exclusive regimes: it remains in the normal basin with probability $1 - p_{\text{crash}}(T)$, or it transitions to the crisis basin with probability $p_{\text{crash}}(T)$. Conditional on the regime, the loss density has a different shape: in the normal basin it is well-approximated by the Cornish-Fisher density centred at $w^{\mathrm{T}}\mu$; in the crisis basin it is centred at the crisis-correlation-induced loss $|\Delta\rho| \cdot \Delta$sensitivity, with little remaining contribution from individual asset skew or kurtosis (because the tail event is dominated by the cross-asset comovement spike).

```
VaRα(T) = (1 − pcrash) · VaRCF(α, T)
          + pcrash · VaRinst(T)

VaRCF(α, T) = zα(S, K) · σw · √T
VaRinst(T)  = |Δρ| · ∂Pw/∂ρ
```

The expression is additive by construction in the partition-function decomposition; we are not patching two estimators heuristically but recognising that the partition function admits this exact split.

3.2 Recovery of Gaussian and classical limits

Two limits provide internal consistency checks:

> **Gaussian limit.** When the portfolio is mean-variance only ($S = K = 0$) and the bistable potential has an arbitrarily high barrier ($S_{\text{inst}} \to \infty$), $p_{\text{crash}} \to 0$ and $\text{VaR}_{\text{CF}} \to z_\alpha \cdot \sigma \cdot \sqrt{T}$. The estimator reduces to RiskMetrics-Gaussian (Longerstaey, 1996).

> **Classical Cornish-Fisher limit.** When the bistable barrier is high but the distribution has non-trivial skew/kurtosis, the estimator reduces to the standard Cornish-Fisher quantile (Cornish & Fisher, 1937), now applied to portfolio-level tensor moments.

> **Crisis-dominated limit.** When S_{inst} is small relative to σ^2—e.g., during a precursor regime in which the correlation matrix has already begun migrating toward the saddle—p_{crash} is $O(1)$ and the estimator is dominated by VaR_{inst}. The perturbative leg loses weight smoothly as the tunneling probability rises.

3.3 Cornish-Fisher tensor moments at the portfolio level

The standardised skew and kurtosis appearing in $z_\alpha(S,K)$ are computed from the portfolio-projected co-moments:

```
S = G³(w, w, w) / (wᵀG²w)^(3/2)
K = G⁴(w, w, w, w) / (wᵀG²w)²  −  3

G²_ij     = E[r_i r_j] − E[r_i] E[r_j]
G³_ijk    = E[r_i r_j r_k] − (mixed lower-order)
G⁴_ijkl   = E[r_i r_j r_k r_l] − (mixed lower-order)
```

The tensors G^3 and G^4 are $O(N^3)$ and $O(N^4)$ respectively in the universe size N. For large universes, computing them on the full holdings is impractical; the standard approach (Martellini & Ziemann, 2010) is to compute the tensors on a held subset (typically <30 assets) and rely on the long-tail-truncation property of the cumulant expansion. Bias-correction follows Friedman (1989).

4. Calibration Protocol

4.1 Implied risk-neutral density

The natural calibration source for σ, the cumulants, and the bistable potential parameters λ, ρ_n, ρ_c is the implied risk-neutral density extracted from option prices (Breeden & Litzenberger, 1978). For a single underlying the second-derivative of the option price with respect to strike recovers the risk-neutral density directly:

$$f_Q(K, T) = \exp(rT) \cdot \partial^2 C(K, T) / \partial K^2$$

For multi-asset settings (Baaquie, 2004) one calibrates the per-asset density and projects through the historical co-cumulant tensor to obtain the portfolio loss density. The bistable potential parameters λ, $\rho_c - \rho_n$ are fitted to the implied density's bimodality (or its absence) using a maximum-likelihood fit at expirations where the implied skew is high ($> 1.5\sigma$ below at-the-money).

4.2 Historical cumulant estimators

When option chains are unavailable or thinly populated, we substitute historical cumulant estimators with bias correction. The unbiased third co-moment estimator is

$$\hat{G}^3_{ijk} = [n / ((n-1)(n-2))] \cdot \Sigma_t \, \delta r_{i,t} \, \delta r_{j,t} \, \delta r_{k,t}$$

with $\delta r = r - \hat{\mu}$ and analogous fourth-order correction for G^4. For the bistable potential parameters in the historical channel, λ is fit to the volatility-of-correlation across rolling 60-day windows and ρ_c is set to the empirical peak correlation observed in identified stress windows (e.g., 2008-Q4, 2020-Q1).

4.3 Time-scale of the partition function

The horizon T entering the estimator is the risk horizon, typically 1 to 10 trading days. The instanton action S_{inst} is computed on the same time scale; the running coupling $\sigma^2(T)$ is obtained from the volatility term structure (Andreev & Howden, 2026b). Cross-scale consistency requires the renormalisation-group treatment summarised in that companion paper.

5. Proposed Empirical Validation

Per Harvey, Liu & Zhu (2016), any new risk estimator should be pre-registered against a breach-rate test before being applied to capital adequacy. We propose the following protocol. A first settlement of the protocol, run 2026-06-11 in its pre-registered historical form (single-asset-class equity book; the frozen thresholds below), is reported in §8.1; the full multi-asset forward protocol remains for a sequel paper.

Pre-registered validation protocol.
(P1) Universe: a fixed large-cap basket and a fixed multi-asset basket (US equity, US Treasury, gold, oil, USD index, EM equity).
(P2) Horizon: 1-day, 5-day, 10-day VaR at $\alpha = 0.95$ and $\alpha = 0.99$.
(P3) Estimators compared: (i) Gaussian/RiskMetrics; (ii) classical Cornish-Fisher; (iii) Cornish-Fisher with bias-corrected G^3 and G^4; (iv) our combined CF-instanton estimator.
(P4) Outcome: number of breaches over a 500-day rolling window. Bias is rejected if the breach rate deviates from α by more than the binomial 99% confidence interval (Kupiec, 1995).
(P5) Threshold: combined estimator must dominate classical Cornish-Fisher on breach-rate proximity to α with $t > 3.0$ over a block-bootstrap with $B = 1000$ and block length 20 days (Harvey-significance).

Caveat. The dilute-instanton-gas approximation breaks down in the regime of strong tunneling (multiple-bounce contributions become important). The single-instanton truncation is appropriate when the typical inter-event spacing is large compared to the bounce timescale, which is the case for monthly-or-longer horizons but may underestimate at sub-daily horizons. Multi-instanton resummation (resurgence; cf. Mariño, 2008) is left for future work and outlined in §7 of this paper.

6. Relation to Bohmian / Stochastic-Mechanical Architecture

Cornish-Fisher is conventionally derived from the Gram-Charlier expansion of a probability density around the Gaussian. In the stochastic-mechanical reformulation of finance (Andreev & Howden, 2026a), the loss density ρ is the projection of a configuration-space density onto the directional axis defined by the portfolio weights. The Cornish-Fisher expansion is then a perturbative truncation of this projected density in moments of the underlying configuration distribution. The non-perturbative correction we add corresponds to the contribution of classical-action saddle points (instantons) of the full configuration-space action.

This identification has two consequences. First, the combined estimator is not a heuristic patch but a structural feature of the underlying density representation. Second, the same instanton calculation feeds the Greek decomposition (Andreev & Howden, 2026c) and the crash-precursor acceleration signal (Andreev & Howden, 2026b); cross-paper consistency requires that the λ, ρ_n, ρ_c parameters be calibrated once and used uniformly.

7. Multi-Instanton Resummation (Future Work)

The dilute-gas truncation retains only the single-bounce sector. For risk horizons long compared to the bounce timescale, multiple instantons contribute and the partition function gains an exponential tower

$$Z(T) = Z_{pert} \cdot \exp[\ T \cdot K_{fluct}^{-1/2} \cdot \exp(-S_{inst} / \sigma^2)\ ]$$

which is the Coleman dilute-gas resummation. This re-exponentiation has two effects: it saturates p_{crash} at high horizon (preventing the linear-in-T divergence of the single-instanton expression), and it introduces a sub-leading two-instanton interference term that may be exploited as a regime-velocity indicator. We leave the detailed empirical study of multi-instanton contributions to a separate paper currently in preparation.

8. Discussion

The combined Cornish-Fisher–instanton estimator inherits the perturbative-cumulant machinery already widely used in financial risk management and adds a non-perturbative correction that activates exactly in the regime where the perturbative series is known to fail. The estimator reduces to Gaussian VaR and to classical Cornish-Fisher in the appropriate limits. In the regime where the implied correlation density approaches bimodality—empirically observable as elevated implied skew at short expirations and rising option-implied correlation volatility—the non-perturbative leg dominates, in line with the historical observation that crisis-regime losses are dominated by correlation comovement spikes rather than by individual-asset tail events.

We emphasise the methodological status of this paper: the combined estimator is a derivation result, calibrated through standard option-implied and historical channels, with a pre-registered breach-rate validation protocol per Harvey–Liu–Zhu (2016). It is not an empirical paper. Empirical results, if any, will be reported as a separate sequel.

8.1 Status: first pre-registered settlement (2026-06-11) and shadow deployment

The first pre-registered settlement of the breach-rate protocol ran on 2026-06-11 as a *historical* primary test under the thresholds frozen at registration (2026-04-20): an equal-weight book of twenty large-capitalisation US equities, daily re-estimation over 2016–2023 with the COVID window (2020-02-19 to 2020-06-30) excluded from breach counting, ten-day 99% VaR, n = 1,909 daily observations, block bootstrap with B = 1000 and block length 20. This is *not* the live shadow ledger: the live shadow accrual (12,910+ records at settlement date) remains unscored and is available for a separate forward registration.

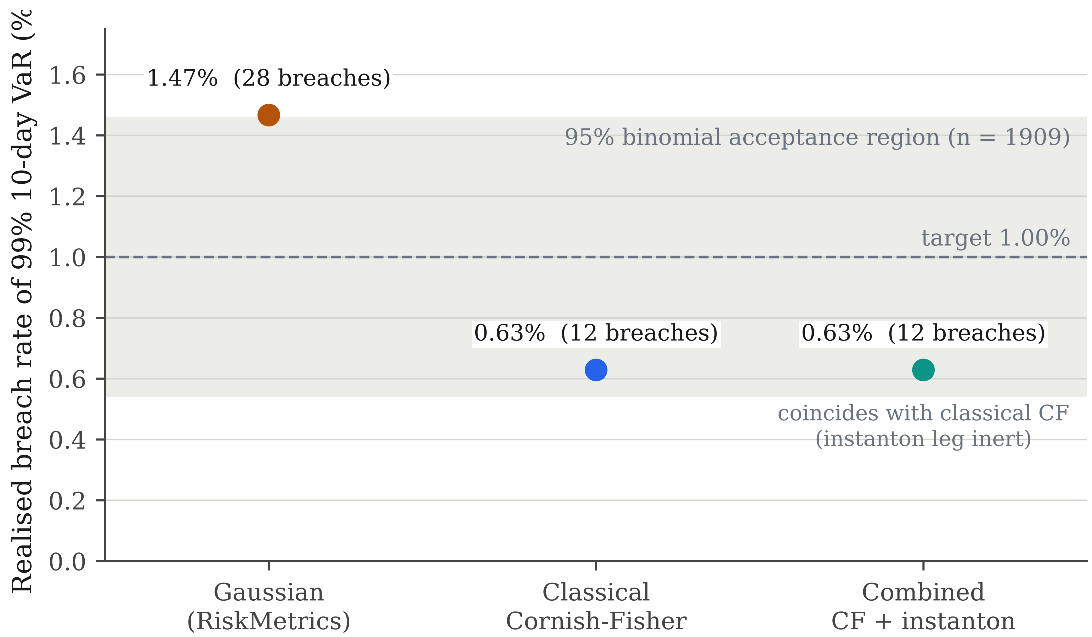

Figure 1. Realised breach rates of the 99% ten-day VaR in the pre-registered historical settlement of 2026-06-11 (equal-weight twenty-stock large-cap book, 2016–2023, COVID window excluded, n = 1,909 daily observations). The shaded band is the 95% binomial acceptance region around the 1% target. The Gaussian benchmark breaches at 1.47% (28 of 1,909), outside the band; the classical Cornish-Fisher quantile breaches at 0.63% (12 of 1,909), inside it. The combined estimator's breach rate coincides exactly with the classical Cornish-Fisher value because the instanton leg was inert throughout the settlement window (see the caveats below). Data: the archived settlement results file (June 2026).

The result is directionally correct but sub-bar. The classical Cornish-Fisher and combined estimators breach at 0.63% (12 of 1,909), inside the 95% binomial acceptance region of the 1% target, where the Gaussian benchmark breaches at 1.47% (28 of 1,909), outside it (Figure 1). The Harvey statistic on the breach-count difference is $t = +2.14$ (block-bootstrap $p = 0.002$), which fails the frozen $t > 3.0$ bar: a genuine sub-threshold result, not a null. The secondary pre-crisis lead-ratio test also fails its frozen bar—mean 1.25 against the required 1.3—with the shortfall heterogeneous by crisis type (Figure 2): the fat-tail spikes (2020 COVID at 1.24, 2023 SVB at 1.50) are led, the persistent 2022 grind (1.00) is not, consistent with higher-moment corrections catching kurtosis spikes rather than slow regime deterioration. Direction is not in question: over the full sample the combined VaR runs about 1.60 times the Gaussian VaR on average, and the pre-crisis lead ratio exceeds the no-lead level 1.0 decisively ($t = 8.2$); the failure is of the frozen lead-ratio magnitude, not of direction.

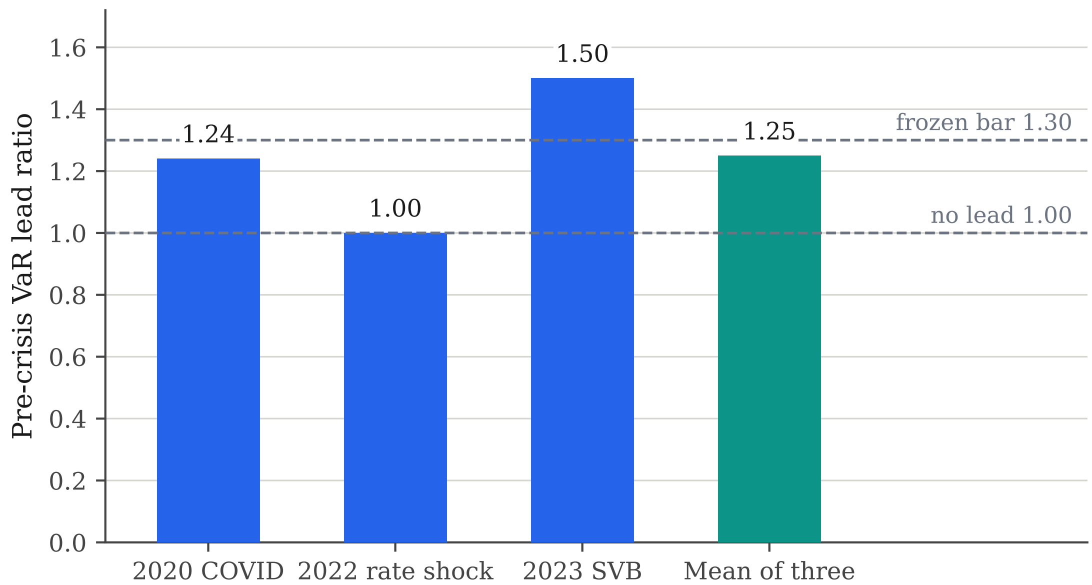

Figure 2. Secondary pre-crisis lead-ratio test from the same settlement: the ratio of the combined estimator to the incumbent classical estimator, averaged over the 30 days preceding each of the three pre-registered stress onsets (March 2020, January 2022, March 2023). Dashed lines mark the no-lead level 1.00 and the frozen acceptance bar 1.30. The mean ratio 1.25 exceeds the no-lead null decisively ($t = 8.2$) but fails the frozen bar; the shortfall is entirely the 2022 grind. Per-window ratios transcribed from the research-register settlement notes; the mean and the test statistic from the archived settlement results file (June 2026).

> **Two honest caveats from the settlement diagnostics.** First, the combined formula collapsed to the pure Cornish-Fisher value in every observation: with the documented default well centres ($\rho_n = 0.25$, $\rho_c = 0.75$) against an average realised correlation of 0.37, the reconstructed tunneling probability averaged 47% and the monotonicity floor left the instanton leg inert—so the settlement validates the perturbative (tensor-moment) leg only, and says nothing yet about the non-perturbative leg this paper contributes. Second, the historical reconstruction and the live pipeline disagree wildly on the tunneling probability (mean 47% reconstructed versus approximately zero live on the settlement date); neither number should be cited as the production calibration until that discrepancy is reconciled. Both findings are recorded in the research register.

Operationally, therefore: position sizing remains on the classical formula (the configuration flag that would route Kelly sizing through the Cornish-Fisher variant stays off); the combined estimator remains a shadow metric, computed and logged every cycle alongside the classical value; the instanton leg is excluded from any VaR role pending recalibration; and promotion requires a new registration with a broader universe or a longer sample for Harvey power. No live risk-management improvement is claimed on the basis of this paper or its first settlement.

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This paper extends the Cornish-Fisher quantile expansion to a partition function with non-perturbative tunneling contributions. The combined estimator recovers Gaussian and classical Cornish-Fisher limits and adds an instanton correction proportional to $\exp(-S_{inst}/\sigma^2)$ in regimes of incipient correlation dislocation.
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